

atmospheric phenomena, one of which we now publish by the kind permission of the President of Columbia University and of Prof. R. S. Woodward, the literary executor of Mr. Cottier. This short paper by Mr. Cottier is especially valuable as indicating the hypotheses or ideas on which his predecessors have based their researches.

By his mental grasp of the complex movements of the air near any obstacle, and his ability to express in rigorous formulas the mechanical reactions that result therefrom, Mr. Cottier gave promise of becoming a remarkably able investigator, and his untimely death was undoubtedly a great loss to meteorology.—C. A.

A SUMMARY OF THE HISTORY OF THE RESISTANCE OF ELASTIC FLUIDS.

By JOSEPH G. C. COTTIER. Dated Columbia University, New York, N. Y., April 27, 1896.

By elastic fluids are understood such fluids as air and other gases, and it is intended to restrict the discussion to such velocities only as are small in comparison to the velocity of sound in the gas. With the exception of ballistic problems and the motion of gases escaping freely from an orifice, almost all ordinary questions fall within this restriction.

Keeping the velocity within these bounds introduces a great simplification in the analysis, for then compressible fluids may, without gross error, be treated as incompressible.

Many writers claim to have discovered that the resistance offered to a moving body by a fluid at rest is not equal to the pressure exerted by a moving fluid on a solid at rest; but the experiments upon which this deduction is based are so unsatisfactory, and the statement itself so improbable, that no allowance has been made in the following essay for such a phenomenon.

The original papers of the writers referred to have been consulted whenever possible; otherwise the authority is given in a footnote.

The history of air resistance may be said to date from the time of Galileo. In his "Discorsi", 1638, he showed that, in consequence of the laws of falling bodies, discovered by him in 1602, the path of a projectile must be parabolic, *if not affected by the resistance of the air*; but his disciples disregarded this injunction, reasoning that a fluid as light as air could not appreciably affect the motion of so heavy a body as a projectile.¹

In 1668-69 a committee of the Royal Academy of Sciences of Paris, consisting of Messrs. Huygens, Mariotte, Picard, and Cassini, made a series of experiments on bodies immersed in currents of water, and from these Huygens deduced the law that the resistance is proportional to the square of the velocity, and also that the pressure on a plane surface is the same as that due to a statical column of the fluid, of height equal to the head due to velocity.

According to Saint Venant,² Pardies showed as early as 1671 that for ships' sails the pressure should be proportional to the $\sin^2 \alpha$, where α has that meaning which will be assigned to it thruout this paper; i. e., it is the angle between the direction of the motion and the plane of the surface, or the complement of the "angle of incidence".

Certain it is, however, that in his "Traite du Mouvement des Eaux", published posthumously in 1686, Mariotte determined the law that resistance is proportional to the square of the velocity, from considerations based on the impact of the molecules of the fluid on the body; and that in the same paper he deduced geometrically the law that the pressure is proportional to the $\sin^2 \alpha$.

¹ Submitted in partial fulfilment of the requirements for the degree of Master of Arts.

² Böhlmann, Hydromechanik, second edition, 1880.

³ B. de Saint Venant, Resistance des Fluides. Published posthumously in the Memoires of the Paris Academy, 1888.

Mariotte died in 1684, and as Newton's "Principia" did not appear until 1687, the credit for the famous laws,

P is proportional to $(\text{Vel.})^2$

and

P is proportional to $\sin^2 \alpha$,

which occur implicitly in Propositions 34 and 35, Book II, of the "Principia", belongs not to Newton, but to Huygens, and to Pardies and Mariotte, respectively.

By some experiments on falling bodies Newton was made aware of the fact that the Huygenian theory of hydrodynamical pressure was not in accordance with practise, and in Proposition 36, Book II, by a process that is unsatisfactory in the extreme, he corrected it so as to give a resultant pressure equal to *one-half* the pressure of a statical column of the fluid of head due to velocity, a result which agreed better with experiment than the first-named law. However, the geometers did not take kindly to Newton's amended theory, but clung to the original Huygenian law.

S'Gravesande, in his work on natural philosophy, 1725, was the first to disagree with Mariotte's or Pardies's law,

P is proportional to $\sin^2 \alpha$,

and to offer the law

P is proportional to $\sin \alpha$.

For small values of α this gives a better result than the former, and was deduced from the consideration that a fluid is not constructed of independent particles, but of a substance that has the property of exerting the same normal pressure in all directions.

Daniel Bernoulli, in 1727, proposed a theory which would have given hydrodynamical pressure equal in amount to the hydrostatical pressure of a column of water of twice the head due to the velocity, but he abandoned this later; and in a memoir published in 1736, making for the first time a distinction between the pressure exerted by an infinite fluid on a body and that due to an isolated jet, he derived that method of treating the latter which has survived to the present day.⁴

Maclaurin's contributions (1742) to this branch of science appear to be confined to the formula for the angle of maximum effort of windmill sails, when P is proportional to $\sin^2 \alpha$.

He found

$$\tan \alpha = \frac{3}{2} \frac{v}{V} + \sqrt{2 + \frac{9}{4} \frac{v^2}{V^2}}$$

where v equals velocity of the vane, and V that of the wind (at right-angles to the first). This is of importance as the first correction to the error in Mariotte's (1686) and Parent's (1704) analysis, which upon the same hypothesis gave α the constant value $55^\circ \pm$, for the effect of the motion of the vane had been neglected.

Robins made a distinct step in advance when in his "New Principles of Gunnery", 1742, he described his apparatus for experimental determination of the resistance of the air, and gave the results of a few tests. This apparatus, the first of its kind, continued much in favor among the later English experimenters. The bodies under observation were fixt at the end of a horizontal arm, rotating about a vertical axis; a falling weight gave the power necessary to keep the arm in motion, and the revolving body itself served the purpose of a governor.

Robins's work was translated into French and annotated by Leonhard Euler. In a note the commentator attempted to obtain a mathematical explanation for the phenomena by summing the components in the direction of motion of the deviating forces necessary to deflect the stream lines from their originally straight path to their disturbed condition. Unfortunately, for a frictionless fluid, such a method gives zero for result, unless the posterior three-quarters of each filament be

⁴ B. de Saint Venant, op. cit.

neglected; and later Euler had to return to the older theory of molecular impact.⁵

D'Alembert's "Nouvelle Theorie de la Resistance des Fluides", 1752, is an important contribution to the science of hydrodynamics. In it may be found a note of an analysis mathematically equivalent to what is now known as "Earnshaw's current function", but altho d'Alembert used complex quantities in his attempts to obtain solutions he found difficulty in integrating the equations which result, and no immediate consequences of his theory followed.

Borda's famous experiments on air resistance were made public in 1763. His apparatus differed from Robins's mainly in having the moving body supported on a vertical arm rotating about a horizontal axis, instead of the reverse. His experiments dealt with small plates, and with prismatic, conical, and ogeeal bodies; from the series of tests on small plates it would appear that

$$P_{90^\circ} \text{ is proportional to } S^{1.1}$$

(P_{90° = total normal pressure, plate exposed at right-angles to direction of wind; S = surface of plate), a law apparently much in favor among the French physicists.

Even as late as 1768, no valid explanation had been offered of the apparent paradox encountered by Euler, for in d'Alembert's "Opuscules" we find him commenting on the peculiar fact that according to the analytical theory of deflected stream lines, a body should be subjected to no resistance, a circumstance which he very kindly "leaves for elucidation to the geometers". However, many experiments had already appeared, and more were soon to follow, that would bring forcibly before the minds of physicists the reality of the resistance encountered.

In 1759, Smeaton communicated to the Royal Society at London that well-known table of wind pressures at different velocities which is generally known by his name. The table is really due to a certain Rouse, a friend of Smeaton's.

If V be the velocity in miles per hour, and P_{90° the normal force in pounds per square foot, Rouse's experiments are well represented by

$$P_{90^\circ} = 0.005 V^2.$$

It is not easy to find who first proposed this formula, but Eytelwein, about 1800, deduced an equivalent formula from Woltmann's and Schoeber's experiments; i. e.,

$$P_{90^\circ} = \frac{4}{3} \frac{S}{2g} V^2$$

where V is the velocity in feet per second, and S is the specific weight of the fluid.⁶

Hutton (as also Rouse) made use of a Robins apparatus when, in 1787, he made the first reliable series of experiments establishing the relation of the air resistance to the angle of exposure, α . If P_α is the intensity of the normal pressure on a plane exposed at an angle α to the wind, Hutton gives

$$P_\alpha = P_{90^\circ} (\sin \alpha)^{1.842 \cos \alpha - 1}$$

Experiments were also made by Hutton on bodies of forms occurring in artillery practise, and the agreement with Borda's results is fairly good.

The experimental work of Vince (London Philosophical Transactions, 1798) brings us to the close of the eighteenth century. Up to this late date no more satisfactory theory of the resistance of fluids had been offered than the Huygenian, that the impinging fluid lost all its momentum upon impact, so that while the face upstream was subjected to hydrodynamic plus hydrostatic pressure, that downstream experienced only the hydrostatic pressure. Of course some writers had combatted this view; Don Georges Juan, in 1771, and

Professor Romme, in 1787, had suggested that the pressure on the downstream side might depend not alone on the hydrostatic pressure of the fluid, but on the difference between it and the hydrodynamic pressure.⁷ Unfortunately, such a hypothesis gave a result about twice as large even as the Huygenian.

Poncelet (Introduction à la Mécanique Industrielle, 1829) offered a new explanation, based on an empirical law deduced by Du Buat from hydraulic experiments made by Messrs. d'Alembert, Condorcet, and Bossut in 1777. This law stated that all the particles of a fluid which are affected in the direction of their motion by the presence of an immersed body, may be included within a cylindrical surface whose axis is parallel to the direction of motion, and whose cross section is 6.46 times the maximum cross section of the body; or that the fluid remains undisturbed at a distance in any direction of about three-quarters ($\frac{3}{4}$) of the diameter of the solid. To Poncelet, then, all problems relating to the resistance of solids to moving fluids reduced themselves to the case of a body suspended centrally in a tube filled with that fluid, with cross section equal to 6.46 times the greatest cross section of the body, and with sides of such material as to offer no frictional resistance to flow. Allowing then for a further contraction of cross section of the jet because of a phenomenon similar in character to the contraction of a free jet, the total drift pressure might be found by computing the change of momentum of the fluid in the normal and the contracted portions of its path.

Altho neither very satisfactory nor withal very fruitful this hypothesis forms the basis of de Saint Venant's extensive memoir already referred to, "Sur la Resistance des Fluides", which was written principally in or about 1847.

About 1825 the Paris Academy of Sciences offered a prize for the best exposition of the theory and practise of the resistance of fluids, which offer was instrumental in bringing to light at least two important contributions to the subject, altho the prize itself was never awarded.

A little later, in 1826, Lieutenant Thibault's results were published; his experiments were made on small planes, 0.327 meter to 0.454 meter on a side, exposed normally to the air on a Borda apparatus. As interpreted by de Louvrié,⁸ these experiments give

$$P_{90^\circ} = 0.115 k_v V^2$$

where V is in meters per second; and P is the net normal pressure in kilograms per square meter.

A memoir by Colonel Duchemin was submitted in 1828 in competition for the above-mentioned prize, receiving "honorable mention". In this memoir will be found the formula for the normal pressure on an inclined plate in terms of that on a plate whose plane is perpendicular to the direction of the wind,

$$P_\alpha = P_{90^\circ} \frac{2 \sin \alpha}{1 + \sin^2 \alpha}$$

This is probably the most reliable formula yet offered, and is generally known as "Duchemin's formula", altho as he offered it there was an additional factor,

$$1 + \frac{\cos^2 \alpha}{15 \sin^2 \alpha}$$

The prize of the Paris Academy having been offered again and again, without bringing any contribution satisfactory to the committee, the prize was withdrawn from competition, and its value awarded to Messrs. Didion, Piobert, and Morin, "à titre d'encouragement". Their paper dealt mainly with the resistance of projectiles; but in it is found an account of experiments made with horizontal planes, 0.25 to 1 square meter in area, falling vertically thru a height of 12 meters,

⁵ B. de Saint Venant, op. cit.

⁶ Young's "Summary of Eytelwein's Hydraulics", in Tredgold's "Tracts on Hydraulics".

⁷ B. de Saint Venant, op. cit.

⁸ Proceedings Chicago Conference of Aerial Navigation, 1893.

with a velocity varying from 0 to 9 meters per second. These gave

$$P_{90^\circ} = \frac{\delta_0}{\delta} (0.036 + 0.084 V^2),$$

where P_{90° is the intensity of normal pressure in kilograms per square meter, δ_0 and δ are the normal and actual specific weights of the air, and V is the velocity in meters per second.

In Germany Professor Schmidt of Göttingen in 1831 had offered the formula

$$R = \beta \cdot \frac{e^{q-1} - q}{q},$$

where $q = \frac{V^2}{2a}$ and α and β are constants.⁹ Unfortunately this paper is not available to me, and the hypothesis upon which it is based is therefore unknown.

A remark by the astronomer Bessel, in 1828, that the time of oscillation of a pendulum was affected by the necessity of moving the circumbient fluid, with a result equivalent to increasing the effective mass of the pendulum, caused Poisson in 1831 to send to the Paris Academy of Sciences an important memoir on the motion of a spherical pendulum in air, in which he attempted to account for both the kinetic and the frictional resistances. George Green, more modest, in 1833 gave the complete solution for the translational motion of an ellipsoid in a frictionless fluid; the more general case of combined rotation and translation was not made public until 1856, by Clebsch.

In 1842, Stokes in England adapted Earnshaw's previously introduced "current function"¹⁰ to solids of revolution moving axially, and with its aid, Stokes in 1850 was able to solve satisfactorily the problem of such a body moving in a viscous fluid, the velocity of the solid being so small, however, that its square is negligible. This problem has engaged the attention of many physicists since that time, among others, Messrs. O. E. Meyer, Oberbeck, R. Hoppe, C. J. H. Lampe, and Bousinesq, with the object of applying the results to the determination of the coefficient of internal friction of fluids.

Passing again to France we find Dupré in 1864 offering the rational formula for resistance—

$$P_a = P \cdot \left(e^{+At^2 \sin^2 \alpha} - e^{-At^2 \sin^2 \alpha} \right),$$

where P_a is the intensity of the normal pressure in kilograms per square meter, P is the aerostatical pressure in kilograms per square meter, and $A = \frac{1.3\rho}{2P} \cdot \frac{T_0}{T}$, ρ being specific mass,

and T and T_0 absolute temperatures.

Von Helmholtz's works in this field are few in number, but, as might be expected from his genius, of the utmost importance. It was from a suggestion contained in one of his memoirs, bearing the date 1873, that M. Thiesen deduced the general form of the equation of resistance,

$$P_{90^\circ} = \rho v^2 l^2 \cdot \varphi \left(\frac{\mu v l}{\rho v^2 l^2} \right),$$

where ρ is the specific mass, v the velocity, l a linear parameter, φ any function of $\frac{\mu v l}{\rho v^2 l^2}$, and μ the coefficient of internal friction.

This theorem is of great value in establishing the relation between the resistances of similar bodies in different gases or in the same gas under different conditions.

⁹ J. C. F. Otto, "Beiträge zur Ermittlung des Luftwiderstandsgesetz," Zeitschrift für Math. und Physik, Vol. II, 1866.

¹⁰ Earnshaw's current function is an expression giving the paths of all the particles of an incompressible, frictionless fluid, moving so that the motion is all parallel to a certain plane, or is "two-dimensional"; and furthermore, so that the motion is "irrotational", one of "pure strain", or possessing a "velocity potential". Such a function has important analytical properties.

Von Helmholtz's most important memoir is that on the theory of "discontinuous motion" in two dimensions, first offered in 1868. Kirchhoff applied this method to the stream lines of a fluid past a plane lamina, without, however, calculating the resultant pressure. Lord Rayleigh, in 1876, independently of any knowledge of Kirchhoff's work, arrived at the result, but pushed his researches to the point of obtaining the pressure per unit length on such an infinitely long plane lamina immersed in an infinite, frictionless fluid. He obtained

$$P_a = \frac{\pi \sin \alpha}{4 + \pi \sin \alpha} \frac{4 + \pi}{\pi} P_{90^\circ}$$

which is perhaps the most satisfactory rational formula yet offered for the resistance of a long, narrow plane exposed obliquely to a current.

In the same memoir will be found an expression for the position of the center of pressure on such a plate. If l is the breadth of the lamina, and d is the distance of the center of pressure from the center of the lamina, we obtain

$$d = \frac{3}{4} \frac{\cos \alpha}{4 + \pi \sin \alpha} l.$$

The best empirical formula for this quantity, that of Jöessel (1870), gives

$$d = (0.3 - 0.3 \sin \alpha) l,$$

which while not exactly agreeable to Lord Rayleigh's, yet offers less discrepancy than might well have been expected, considering the difference of the conditions.

Bobyleff in 1881 applied the theory of discontinuous motion to obtain the resultant drift on a wedge formed of two planes, each of breadth l , inclined at an angle of 2α to each other and at an angle α to the direction of the current. (See fig. 1.)

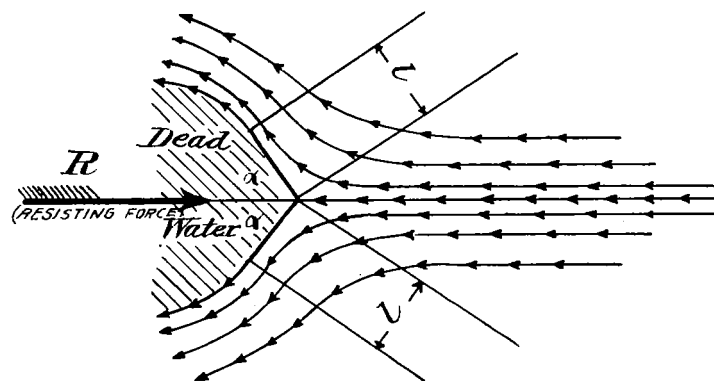


FIG. 1.—Motion of a fluid past a wedge.

In this case,

$$R = \rho V^2 S \frac{2a^2}{\pi \lambda},$$

where

$$\lambda = 1 + \frac{2a}{\pi} + \frac{4a^2}{\pi^2} \int_0^1 \frac{x^{-a/\pi}}{1+x} dx,$$

ρ = specific mass, V = velocity, S = area = $2lh$, and $2a$ = the angle of the planes.¹¹

Owing to the impossibility of formation of such a surface as is here supposed, this theory and the results obtained by it have been criticized by many. Some sort of surface of discontinuity may of course be formed, and Lord Kelvin in 1887 offered his theory of "coreless vortices" as being nearer the physical conditions.

Among the various rational formulas proposed since Lord Rayleigh's may be mentioned that of Professor Ferrel,

$$\log_{10} \left(1 + \frac{P_{90^\circ}}{P} \right) = \frac{V^2}{360,940} \frac{T_0}{T},$$

¹¹ Lamb, "Hydromechanics", 1895.

where P_{90° is the intensity of the net normal pressure, P is the aerostatical pressure, V is the velocity in meters per second, and T and T_0 are absolute temperatures.

This formula does not rest on a sound basis, for it may be derived from that given by Lord Rayleigh, by expanding and rejecting the higher powers of $\frac{V}{a}$ than the square

$$P_{90^\circ} = P \left\{ \left(1 + \frac{r-1}{2} \frac{V^2}{a^2} \right)^{\frac{r}{r-1}} - 1 \right\}$$

for the resistance that would be encountered if the impinging filament of fluid could be supposed to disappear absolutely, after imparting all of its momentum to the plate; (r is here equal to the ratio of the specific heats of the gas at constant pressure and at constant volume, and a is the velocity of sound in the gas).

E. Toepler¹² in 1887 proposed a formula based on considerations derived from the molecular theory of gases:

$$P_{90^\circ} = 4 P \frac{V}{\Omega},$$

where, as before, P is the aerostatical pressure, V the velocity of the plate, and Ω the mean molecular velocity of the particles of the gas. Experiments made by G. A. Hirn, in 1882, appear, however, to disprove any such immediate dependence of the resistance on the temperature as is here implied, when the density remains constant.

Ch. de Louvri  s formula (1890),

$$P_a = \frac{2 \sin \alpha (1 + \cos \alpha)}{1 + \cos \alpha + \sin \alpha} P_{90^\circ}$$

is quite satisfactory. The basis for the physical considerations on which this rational formula is founded may be discovered in Colonel Duchemin's experiments.

The latest addition to this collection of formulas, that of Lord Kelvin (1894), requires a word of explanation.

The resistance experienced by a moving solid in a "perfect" or frictionless fluid would be zero, if no surface of discontinuity were formed, or if the fluid obeyed the so-called "electrical law" of flow, requiring under certain conditions an infinite tension to be resisted. The kinetic energy of the body would, however, be changed by the presence of the fluid, and the additional kinetic energy is found to be

$$T = \frac{\pi \rho a^2 v^2}{2}$$

per unit of length of an infinitely long lamina of breadth a ; or

$$T = \frac{4}{3} \rho c^3 v^3$$

for a circular plate of radius c , ρ in both cases being the specific mass of the fluid, and v the velocity in a direction normal to the plate.¹³

These results were obtained by supposing the minor axis of an elliptical cylinder, and the shorter axis of a prolate spheroid, respectively, to become equal to zero. The motion of the fluid is in both cases irrotational, and therefore in the first case, for an infinitely long lamina, it could have been generated by an impulsive pressure of

$$F = \frac{\pi}{4} \rho a^2 v$$

per unit length. From this value of the impulsive pressure and from the assumption of a velocity in its own plane of u , that is large compared with v , Lord Kelvin found the resistance to be

$$P_a = \frac{\pi}{2} \rho u v$$

which is equivalent to

$$P_a \text{ is proportional to } \sin \alpha \cos \alpha V^2$$

for α small, and the length great compared to the breadth in the direction of motion.

A brief account of the most notable series of experiments since 1870 must close this summary.

The measurements of G. H. L. Hagen (1874) have become classic; they may be expressed by

$$P_{90^\circ} = (0.00707 + 0.0001125 p) V^2,$$

where P_{90° is the intensity of normal pressure in grams per square decimeter, V is the velocity in decimeters per second, and p is the perimeter in decimeters. Unfortunately this formula can not be safely applied to plates of more than 20 centimeters on a side.

L. de Saint Loup (1879), for a plate 10 by 20 centimeters, found

$$P_a = 0.1768 (4 \sin \alpha - 1) (11 V + 1.061 V^2)$$

where P is the pressure in grams per square decimeter, and V is the velocity in meters per second.

From the above-mentioned experiments of G. A. Hirn, in 1882, and from the carefully executed tests of Messrs. Cailletet and Colardeau in 1893, it appears definitely settled that, even for different gases, the resistance is not directly affected by the temperature, but only indirectly thru the resulting change of density, and that this resistance is directly proportional to the density of the gas and to the square of the velocity of the vane.

Otto Lilienthal, in 1889, experimented on the resistance of curved vanes, but without arriving at a satisfactory general formula.

Lieutenant Crosby in 1890 published an account of a series of experiments purporting to show that the resistance of the air was directly proportional to the velocity instead of to its square, but these experiments are not viewed with much favor.

Mr. W. H. Dines' extensive tests, also in 1890, on small plates exposed both normally and at an angle to the wind, give

$$P_{90^\circ} = 0.0029 V^2$$

where P_{90° is the pressure in pounds per square foot, and V is the velocity in miles per hour; the measurements with the plate exposed obliquely have not been embodied in a formula.

This list is fittingly closed by a mention of Mr. S. P. Langley's very satisfactory experiments, published in 1891. His most refined apparatus gave, as the probable value of the normal pressure P_{90° on a plate exposed at right-angles to the direction of the wind, on planes of from 6 to 12 inches on a side

$$P_{90^\circ} = 0.0087 \frac{\delta}{\delta_0} V^2,$$

where P_{90° is the pressure in grams per square centimeter, V is the velocity in meters per second, δ is the specific weight of the air at the time of the experiment, and δ_0 is that for a pressure of 760 millimeters mercury and at a temperature of 10° centigrade. Mr. Langley's experiments on planes exposed at an angle to the current of air agree so nearly with Colonel Duchemin's formula,

$$P_a = \frac{2 \sin \alpha}{1 + \sin^2 \alpha} P_{90^\circ},$$

that no new one is offered.

LOCAL FORECASTING AT ESCANABA.

By W. P. STEWART, Observer, Weather Bureau. Dated Escanaba, Mich., August 31, 1907.

Aside from, or rather superimposed upon, the more or less regular sequence of weather changes due to passing cyclones and anticyclones, most localities have a system of minor variations caused by local peculiarities of topography or location with regard to neighboring bodies of water, etc. In some cases these minor variations become so pronounced as greatly to modify the current weather of the region. Probably in no portion of the United States is this more noticeable than in the upper Lake region. The water of the Lakes, relatively

¹² Wiedemann's "Beibl  tter" Vol. XI, 1887, p. 747. ¹³ Lamb, op. cit.